On the k-tuple domination number of graphs*

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Abstract

Let $k \geq 1$ be an integer and G be a graph of minimum degree $\delta(G) \geq k-1$. A set $D \subseteq V(G)$ is said to be a k-tuple dominating set of G if $|N[v] \cap D| \geq k$ for every vertex $v \in V(G)$. The minimum cardinality among all k-tuple dominating sets is the k-tuple domination number of G. In this work, we present new tight bounds on the k-tuple domination number of a graph. Some bounds can also be seen as relationships between this parameter and several other domination parameters in graphs.

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Given a vertex $v \in V(G)$, $N(v)$ denotes the *open neighbourhood* of v in G. Given a vertex $v \in V(G)$ and a set $D \subseteq V(G)$, we denote by $\deg_D(v) = |N(v) \cap D|$ the number of vertices in D adjacent to v and let $\deg_D[v] = |N[v] \cap D|$. The *minimum* and *maximum degrees* of G will be denoted by $\delta(G)$ and $\Delta(G)$, respectively.

Fink and Jacobson [5, 6] introduced the concept of k-domination in graphs. Given a graph G, a set $D \subseteq V(G)$ is said to be a *k-dominating set* of G if $\deg_D(v) \geq k$ for every $v \in V(G) \setminus D$. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality among all k-dominating sets of G.

Harary and Haynes [8, 9] introduced the concept of k-tuple domination in graphs. Given a graph G and a positive integer $k \leq \delta(G)+1$, a set $D \subseteq V(G)$ is said to be a *k-tuple dominating set* of G if $\deg_D[v] \geq k$ for every $v \in V(G)$. Observe that the 1-tuple dominating set of G is the same as the dominating set of G. The *k-tuple domination number* of G, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality among all k-tuple dominating sets of G. For a comprehensive survey on k-tuple domination in graphs, we suggest the chapter [7] due to Hansberg and Volkmann. In addition, some recent results on this parameter can be found in [1–3, 10, 11].

In this work, we present new tight bounds on the k-tuple domination number of a graph. Some bounds can also be seen as relationships between this parameter and several other domination parameters in graphs.

2 Results

Hansberg and Volkmann [7] put into context all relevant research results on k-tuple domination that have been found up to 2020. In that chapter, they posed the following open problem.

*The results presented in this work have been published in [2] and [3].

Problem 2.1. *(Problem 5.8, p. 194, [7]) Give an upper bound for $\gamma_{\times k}(G)$ in terms of $\gamma_k(G)$ for any graph G of minimum degree $\delta(G) \geq k-1$.*

The following theorem provides a solution to the previous problem.

Theorem 2.2. *Let $k \geq 2$ be an integer. For any graph G with $\delta(G) \geq k-1$,*

$$\gamma_{\times k}(G) \leq k\gamma_k(G) - (k-1)^2.$$

The next theorem provides a new upper bound for the k-tuple domination number of a graph G in terms of the k-domination number and the k'-tuple domination number ($k' \in \{1, \ldots, k-1\}$).

Theorem 2.3. *Let k', k be two integers such that $k > k' \geq 1$. For any graph G with $\delta(G) \geq k-1$,*

$$\gamma_{\times k}(G) \leq \gamma_{\times k'}(G) + (k-k')\gamma_k(G).$$

The following result is an immediate consequence of Theorem 2.3.

Corollary 2.4. *Let $k \geq 2$ be an integer. For any graph G with $\delta(G) \geq k-1$,*

(i) $\gamma_{\times k}(G) \leq (k-1)\gamma_k(G) + \gamma(G)$.

(ii) $\gamma_{\times k}(G) \leq \gamma_{\times (k-1)}(G) + \gamma_k(G)$.

Favaron et al. [4] showed that $\gamma_{\times k}(G) \geq \gamma_{k'}(G) + k - k'$ for any graph G with $\delta(G) \geq k > k' \geq 1$. The following result improves the previous one whenever $diam(G) \geq 5$.

Theorem 2.5. *Let k', k be two integers such that $k > k' \geq 1$. For any connected graph G with $\delta(G) \geq k$,*

$$\gamma_{\times k}(G) \geq \gamma_{k'}(G) + (k-k')\left\lceil \frac{diam(G)+1}{5} \right\rceil.$$

Harary and Haynes [9] showed that $\gamma_{\times k}(G) \geq \frac{2kn-2m}{k+1}$ for any graph G of order n and size m with $\delta(G) \geq k-1$. The following result improves the bound above for any graph with minimum degree at least k.

Proposition 2.6. *Let $k \geq 2$ be an integer. For any graph G of order n and size m with $\delta(G) \geq k$,*

$$\gamma_{\times k}(G) \geq \frac{(\delta(G)+k)n - 2m}{\delta(G)+1}.$$

The following result provides a new upper bound on $\gamma_{\times k}(G)$ in terms of the order and the maximum degree of a graph G with minimum degree $\delta(G) \geq k$.

Proposition 2.7. *Let $k \geq 2$ be an integer. For any graph G of order n with $\delta(G) \geq k$,*

$$\gamma_{\times k}(G) \leq \frac{k\Delta(G)}{k\Delta(G)+1}\, n.$$

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