Domination in Colored Graphs

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Abstract

We introduce a natural concept of domination for a colored graph which leads to two optimization problems.

1 Introduction and preliminaries

As it is well–known, the domination number $\gamma(G)$ of a graph G is the size of a smallest set D of vertices of G such that every vertex outside D has at least one neighbor in D, that is, D is a dominating set of G. Additionally, there are several papers dealing with the problem of domination on colored graphs (see [1, 2]). In those papers, the definition is too restrictive for some applications and their authors consider that a *domination coloring* of a graph G is a coloring of G such that each vertex of G dominates at least one color class, and each color class is dominated by at least one vertex.

However, it is natural to think an ordering in any domination relationship. In this way, we introduce a natural concept of domination in the context of colored graph. Throughout this work, a graph is finite, undirected and simple, and is denoted by $G(V, E)$, where V and E are its vertex-set and edge-set, respectively.

Given a graph $G(V, E)$, a coloring of G is a map $c : V \longrightarrow \{0, 1, 2 \dots\}$ such that $c(v_i) \neq c(v_j)$ if v_i and v_j are adjacent vertices. If the coloring uses exactly k colors, it is referred as a k–coloring, and, as it is well known, the chromatic number $\chi(G)$ of a graph G is the minimum value of k. Hereinafter, an optimal coloring is a $\chi(G)$–coloring.

Then, any coloring c of $G(V, E)$ provides a natural (partial) ordering of V, and so the pair (G, c) can be considered as an acyclic digraph such that every edge starts from its vertex with the higher color. In this directed context, $\delta^-(u)$ and $N^-(u)$ denote the in–degree and in–neighborhood of the vertex u, respectively, and, analogously, $\delta^+(u)$ and $N^+(u)$ are the out–degree and out–neighborhood of u, respectively. By language abuse, we will work with these concepts over graphs which are provided of colorings.

Finally, given a graph G with a coloring c, we say that $D \subseteq V$ is an *up–color dominating c–set* of the pair (G, c) if: (1) for any vertex v not in D there exists an adjacent vertex $d \in D$ such that $c(v) < c(d)$; (2) D contains no vertex of color 0. Thus, c and D establish a dominant–submissive relationship on G, in which the vertices v and d are *submissive* and *dominant*, respectively.

Previous definitions allow us to define two different optimization goals. In this way, following the tradition, we can minimize the size of the dominating set and thus the smallest cardinal of an up–color dominating c–set for the pair (G, c) is called *the up–color domination c–number* of (G, c) and it is denoted by $\gamma_{uc}(G, c)$.

On the other hand, as far as the coloring gives in a natural way a weight to the vertices of the graph (and to subsets of the vertices, adding the individual weights), the minimum weight of an up–color dominating c–set for the pair (G, c) is called *the up–color domination c–weight* of (G, c) and it is denoted by $\omega_{uc}(G, c)$. The vertex sets that lead to $\gamma_{uc}(G, c)$ and $\omega_{uc}(G, c)$ are called *γ_{uc}–dominating c-set* and *ω_{uc}–dominating c-set*, respectively. Obviously, both any γ_{uc}– and ω_{uc}–dominating c-set contains all vertices colored with the highest color of c.

Furthermore, the minimum $\omega_{uc}(G, c)$ among all possible colorings c is called *the up–color domination weight of G* and denoted by $\Omega_{uc}(G)$. Every coloring c verifying $\Omega_{uc}(G) = \omega_{uc}(G, c)$ is named *Ω_{uc}–dominating coloring* and the associated ω_{uc}–dominating c-set is called *Ω_{uc}–dominating set.*

2 Up–color domination number

The definition of up–color domination number makes sense: consider $K_{2,3}$, if we assign to the class with three elements the largest number in a 2–coloring c, then $\gamma_{uc}(K_{2,3}, c) = 3$, but if we interchange the colors, then $\gamma_{uc}(K_{2,3}, c') = 2 = \gamma(G)$.

2.1 Computing $\gamma_{uc}(G, c)$

Given a pair (G, c), a vertex $v \in V(G)$ is a *local maximum* for c if $c(u) \leq c(v)$ for all neighbor u of v. Let $M_c(G)$ denote the set of local maxima of G for c (obviously, $c^{-1}(i) \subseteq M_c(G)$). Now:

Lemma 2.1. *Given a pair (G, c), $\gamma_{uc}(G, c) \geq \max\{|M_c(G)|, \gamma(G)\}$.*

As an immediate consequence of Lemma 2.1 we can find examples of pairs (G, c) such that $\gamma_{uc}(G, c) > \gamma(G)$.

For the complexity, we can define the problems of computing $\gamma_{uc}(G, c)$ and $\omega_{uc}(G, c)$ in the following way:

UP–COLOR DOMINATION C–NUMBER
INSTANCE: A graph $G = (V, E)$, a coloring c and a positive integer k.
QUESTION: Is $\gamma_{uc}(G, c) \leq k$?
UP–COLOR DOMINATION C–WEIGHT
INSTANCE: A graph $G = (V, E)$, a coloring c and a positive integer k.
QUESTION: Is $\omega_{uc}(G, c) \leq k$?

It is trivial to prove that if c uses only two colors (1 and 2), then the ω_{uc}–dominating c-set is just $c^{-1}(2)$. Thus, using only two colors $\gamma_{uc}(G, c)$ and $\omega_{uc}(G, c)$ can be computed in linear time. That is not the case if c uses at least three colors, even if G is a bipartite graph.

Theorem 2.2. UP–COLOR DOMINATION C–NUMBER *and* UP–COLOR DOMINATION C–WEIGHT *are NP–complete problems, even if the graph is bipartite and the image of the coloring is $\{1, 2, 3\}$, and even for optimal coloring.*

On the other hand, we have:

Proposition 2.3. *For any graph G there exists a coloring c such that $\gamma_{uc}(G, c) = \gamma(G)$.*

The smallest number of colors $\chi_{uc}(G)$ needed to preserve $\gamma_{uc}(G,c) = \gamma(G)$ will be called *the chromatic up–color domination number* of G. As a direct consequence of the proof of Proposition 2.3, we obtain:

Corollary 2.4. *For any graph G, $\chi(G) \leq \chi_{uc}(G) \leq 2\chi(G) - 1$.*

CHROMATIC UP–COLOR DOMINATION NUMBER
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Is $\chi_{uc}(G) \leq k$?

Theorem 2.5. CHROMATIC UP–COLOR DOMINATION NUMBER *is an NP–hard problem. Even for $k = 3$. But if we restrict to trees, the problem can be solved in linear time.*

3 Up–color domination weight

One of the aims of the up–color domination problem in graphs is to minimize the sum of the vertex weight in up–color dominating sets. We recall that an Ω_{uc}–dominating coloring of a graph G is a coloring such that $\omega_{uc}(G,c) = \Omega_{uc}(G)$. Then, given any graph, a first natural question is whether all Ω_{uc}–dominating colorings use the same number of colors. The bipartite graph $K_{3,3}$ provides a simple example of the negative answer to this question.

In addition, since the Grundy number of $K_{3,3}$ is two, it is concluded that not all Ω_{uc}–dominating colorings can be obtained applying the greedy coloring algorithm.

A first result, with some bounds is:

Proposition 3.1. *For any graph,*

$$\max\{\chi(G) - 1, \gamma(G)\} \leq \Omega_{uc}(G) \leq \min\{\frac{3}{2}(\chi(G) - 1)\gamma(G), i(G)\chi(G)\} \leq \frac{1}{4}n^2.$$

And $\chi(G) - 1 = \Omega_{uc}(G)$ if and only if G is a cone graph (the closed neighborhood of one vertex is the whole graph).

From the point of view of the complexity, we define:
UP–COLOR DOMINATION WEIGHT
INSTANCE: A graph $G = (V, E)$ and a positive integer k.
QUESTION: Is $\Omega_{uc}(G) \leq k$?

Theorem 3.2. UP–COLOR DOMINATION WEIGHT *is an NP–complete problem. But it can be solved in polynomial time in the case of trees.*

References

[1] S. Aramugam, J. Bagga and K. Chandrasekar, *On dominator colorings in graphs*, Proceedings-Mathematical Sciences 122(4), (2012) 561–571.

[2] Maheswari, A Uma and Samuvel, J Bala *Power dominator chromatic number for some special graphs*, Int. J. Innov. Technol. Explor. Eng 8(22), 2019, 3957–3960.