Double domination in rooted product graphs

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Abstract

A set D of vertices of a graph G is a double dominating set of G if $|N[v] \cap D| \geq 2$ for every $v \in V(G)$, where $N[v]$ represents the closed neighbourhood of v. The double domination number of G is the minimum cardinality among all double dominating sets of G. In this article, we show that if G and H are graphs with no isolated vertex, then for any vertex $v \in V(H)$ there are six possible expressions, in terms of domination parameters of the factor graphs, for the double domination number of the rooted product graph $G \circ_v H$. Additionally, we characterize the graphs G and H that satisfy each of these expressions.

1 Introduction

Domination in graphs is well studied in graph theory and the literature on this subject has been surveyed and detailed in the books [1,2]. In [3], Harary and Haynes extended the idea of domination in graphs to the more general notion. They introduced the concept of double domination in graphs and, more generally, the concept of k-tuple domination. Given a graph G of minimum degree $\delta(G)$ and a positive integer $k \leq \delta(G) + 1$, a set $D \subseteq V(G)$ is said to be a *k-tuple dominating set* of G if $|N[v] \cap D| \geq k$ for every vertex $v \in V(G)$, where $N[v]$ represents the closed neighbourhood of v. The *k-tuple domination number* of G, denoted by $\gamma_{\times k}(G)$, is the minimum cardinality among all k-tuple dominating sets of G. A $\gamma_{\times k}(G)$-set is a k-tuple dominating set of cardinality $\gamma_{\times k}(G)$. The cases $k = 1$ and $k = 2$ correspond to domination and double domination, respectively. In such a case, $\gamma(G)$ and $\gamma_{\times 2}(G)$ denote the domination number and the double domination number of graph G, respectively.

For any graph G, a set D is a *2-dominating set* of G if $|N(v) \cap D| \geq 2$ for every vertex $v \in V(G) \setminus D$, where $N(v)$ represents the open neighbourhood of v. The *2-domination number* of G, denoted by $\gamma_2(G)$, is the minimum cardinality among all 2-dominating sets of G.

The double domination in graphs has been extensively studied. In [4], Hansberg and Volkmann put into context all relevant research results on double domination that have been found up to 2020. In addition, we suggest the recent papers [5–8]. However, this classic domination parameter has not yet been studied in rooted product graphs. With this work we pretend to solve this gap in the theory.

2 Results

Let G and H be two graphs with no isolated vertex and v be a vertex of H, which we called *root vertex*. The *rooted product graph* $G \circ_v H$ is defined as the graph obtained from G and H by taking one copy of G and $n(G)$ copies of H and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H for every $i \in \{1, 2, \ldots, n(G)\}$.

We now proceed to introduce a new domination parameter which plays an important role in one of the possible values that the double domination number of the rooted product graphs takes.

Let $A, B \subseteq V(G)$. We say that an ordered pair (A, B) of disjoint sets A and B is a *quasi-double dominating pair* of G if $A \cup B \in \mathcal{D}_2(G)$ and $B \in \mathcal{D}_{\times 2}(G - A)$. The *quasi-double domination number* of G, denoted by $\gamma_{q\times 2}(G)$, is defined to be

$$\gamma_{q\times 2}(G) = \min\{|A| + |B| : \, A \cup B \in \mathcal{D}_2(G) \text{ and } B \in \mathcal{D}_{\times 2}(G - A)\}.$$

Theorem 2.1. *Let G and H be two graphs with no isolated vertex. If $v \in V(H)$, then*

$$\gamma_{\times 2}(G \circ_v H) \in \left\{ \begin{array}{l} n(G)\gamma_{\times 2}(H), \\ \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 1), \\ \gamma_2(G) + n(G)(\gamma_{\times 2}(H) - 1), \\ \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1), \\ n(G)(\gamma_{\times 2}(H) - 1), \\ \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2) \end{array} \right\}.$$

Now, we proceed to show some simple examples in which we can observe that the expressions of $\gamma_{\times 2}(G \circ_v H)$ given in Theorem 2.1 are realizable.

Example 1. *Let G be a graph with no isolated vertex. If H is C_4, C_5 or one of the graphs shown in Figure 1, then the resulting values of $\gamma_{\times 2}(G \circ_v H)$ for some specific roots are described below.*

(i) $\gamma_{\times 2}(G \circ_v C_4) = 3n(G) = n(G)\gamma_{\times 2}(C_4)$.

(ii) $\gamma_{\times 2}(G \circ_v C_5) = 3n(G) = n(G)(\gamma_{\times 2}(C_5) - 1)$.

(iii) $\gamma_{\times 2}(G \circ_v H_1) = \gamma_{\times 2}(G) + 2n(G) = \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H_1) - 2)$.

(iv) $\gamma_{\times 2}(G \circ_v H_2) = \gamma(G) + 2n(G) = \gamma(G) + n(G)(\gamma_{\times 2}(H_2) - 1)$.

(v) $\gamma_{\times 2}(G \circ_v H_3) = \gamma_2(G) + 2n(G) = \gamma_2(G) + n(G)(\gamma_{\times 2}(H_3) - 1)$.

(vi) $\gamma_{\times 2}(G \circ_v H_4) = \gamma_{q\times 2}(G) + 2n(G) = \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H_4) - 1)$.

From now on, we show the results that allow us to characterize the graphs G, H, as well as the root $v \in V(H)$ that satisfy each of the six expressions given in Theorem 2.1.

Theorem 2.2. *If $\gamma_{\times 2}(G) < n(G)$, then the following statements are equivalent.*

(i) $\gamma_{\times 2}(G \circ_v H) = \gamma_{\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 2)$.

(ii) $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$.

Theorem 2.3. *Let G and H be two graphs with no isolated vertex and $v \in V(H)$. Then $\gamma_{\times 2}(G \circ_v H) = n(G)(\gamma_{\times 2}(H) - 1)$ if and only if one of the following conditions holds.*

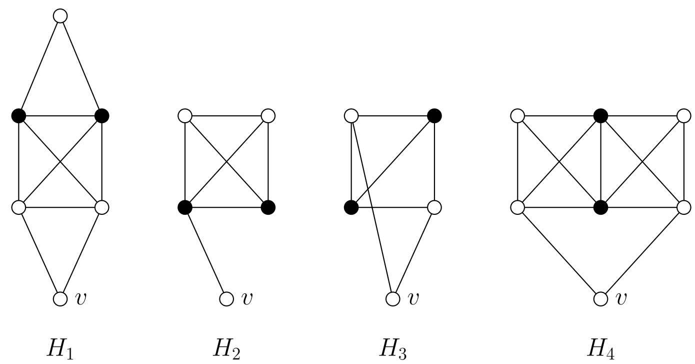

Figure 1: The set of black-coloured vertices forms a $\gamma_{\times 2}(H_i - v)$-set, for $i \in \{1, 2, 3, 4\}$.

(i) $\gamma_{\times 2}(G) = n(G)$ *and* $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 2$.

(ii) $\gamma_{\times 2}(H - v) \geq \gamma_{\times 2}(H) - 1$ *and there exists a set* $W \subseteq V(H) \setminus N[v]$ *of cardinality* $\gamma_{\times 2}(H) - 2$ *which is both a DDS of* $H - N[v]$ *and a TDS of* $H - v$.

Theorem 2.4. *Let* G *be a graph such that* $\gamma_{\times 2}(G) < n(G)$. *Let* H *be a graph with no isolated vertex and* $v \in V(H)$. *Then* $\gamma_{\times 2}(G \circ_v H) = n(G)\gamma_{\times 2}(H)$ *if and only if either* $v \in \mathcal{S}(H)$ *or the following conditions hold.*

(i) $\gamma_{\times 2}(H - v) \geq \gamma_{\times 2}(H)$.

(ii) *If* $H - N[v]$ *has no isolated vertices, then every subset of* $V(H) \setminus N[v]$ *of cardinality* $\gamma_{\times 2}(H) - 2$ *is not a DDS of* $H - N[v]$ *or it is not a TDS of* $H - v$.

Theorem 2.5. *Let* G *and* H *be two graphs with no isolated vertex and* $v \in V(H)$. *Then* $\gamma_{\times 2}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\times 2}(H) - 1)$ *if and only if the following conditions hold.*

(i) $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 1$ *and one of the following conditions holds.*

  (a) $\gamma(G) = \gamma_2(G)$ *and there exists a* $\gamma_{\times 2}(H)$*-set containing the vertex* v.

  (b) $\gamma(G) < \gamma_2(G)$ *and there exists a* $\gamma_{\times 2}(H - v)$*-set* W *such that* $N(v) \cap W \neq \varnothing$.

(ii) *If* $H - N[v]$ *has no isolated vertices, then every subset of* $V(H) \setminus N[v]$ *of cardinality* $\gamma_{\times 2}(H) - 2$ *is not a DDS of* $H - N[v]$ *or it is not a TDS of* $H - v$.

Theorem 2.6. *Let* G *be a graph with no isolated vertex such that* $\gamma(G) < \gamma_2(G) < \gamma_{q\times 2}(G)$. *Let* H *be a graph with no isolated vertex and* $v \in V(H)$. *Then* $\gamma_{\times 2}(G \circ_v H) = \gamma_2(G) + n(G)(\gamma_{\times 2}(H) - 1)$ *if and only if the following conditions hold.*

(i) $\gamma_{\times 2}(H - v) = \gamma_{\times 2}(H) - 1$.

(ii) $N(v) \cap W = \varnothing$ *for every* $\gamma_{\times 2}(H - v)$*-set* W.

(iii) *There exists a* $\gamma_{\times 2}(H)$*-set containing the vertex* v.

(iv) *If* $H - N[v]$ *has no isolated vertices, then every subset of* $V(H) \setminus N[v]$ *of cardinality* $\gamma_{\times 2}(H) - 2$ *is not a DDS of* $H - N[v]$ *or it is not a TDS of* $H - v$.

3 Conclusions

Theorem 2.1 shows there are six possible expressions for the double domination number in rooted product graph $\gamma_{\times 2}(G \circ_v H)$. Moreover, we have been able to characterize the graphs G and H, and the root $v \in V(H)$, that satisfy five of these expressions through the Theorems 2.2, 2.3, 2.4, 2.5 and 2.6. For the case of the equality $\gamma_{\times 2}(G \circ_v H) = \gamma_{q\times 2}(G) + n(G)(\gamma_{\times 2}(H) - 1)$, the corresponding characterization can be derived by eliminating the previous ones from the family of all graphs G and H with no isolated vertices and roots v of H.

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