Semi-cubic graphs and cages of even girth*

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Abstract

In this work, we study a generalization of the *Cage Problem*. We construct families of semi-cubic graphs with fixed girth and few vertices with two different techniques, one of them consists in identifying remote vertices (it is also used in different papers cited in this work to construct bi-regular graphs), and the second consists in use some specific voltage graphs. Some of these graphs are the smallest that exist because attain the lower bounds given previously.

1 Introduction

The graphs that we study in this work are simple and finite. The *girth* of a graph is the length of the smallest cycle in the graph and a *semi-cubic graph* is a graph in which all vertices have degrees either m or 3. An $(3,m;g)$*-semi-cubic graph* is a semi-cubic graph of fixed girth g, and a $(3,m;g)$-*semi-cubic cage* is a $(3,m;g)$-semi-cubic graph with the smallest possible number of vertices. In this work, we construct families of semi-cubic graphs for infinitely many values of m and even girth $g = \{6, 8, 10, 12\}$. The cases $g = \{6, 8\}$ are semi-cubic cages, including the unique 6-cage and 8-cage when $m = 3$.

This problem is a generalization of the *Cage Problem*, where an (r, g)*-cage* is a r-regular graph of fixed girth g; with the smallest possible number of vertices among all (r, g)-graphs (r-regular graphs of girth g). There exists a simple lower bound due to Moore; an (r, g)-graph that attains this lower bound is called a *Moore cage*. The Cage Problem consists of finding (r, g)-cages; or, as these graphs exist for very limited sets of parameters (r, g), it consists of finding (r, g)-graphs with few vertices (for more information of this problem see [8]).

A generalization of this problem has been studied for *biregular graphs*, denoted as $(r,m;g)$*-graphs*, which generalize (r, g)-graphs by allowing vertices of two degrees, with cages generalizing to *biregular*

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cages. Specifically, given three positive integers $\{r, m, g\}$ with $2 \leq r < m$ and $g \geq 3$ an $(r, m; g)$-*cage* is a graph in which all vertices have degrees either r or m, has girth g and minimum order (it has the smallest possible number of vertices among all $(r, m; g)$-graphs). We denote the order of a $(r, m; g)$-biregular cage as $n(r, m; g)$. Biregular graphs have been studied by many authors since Chartrand, Gould and Kapoor proved their existence in [7].

2 Results

We construct families of biregular graphs of small order or biregular cages for fixed girth g when $r = 3$ and $m > 3$, that is families of semi-cubic graphs of small order or cages of even girth using two different techniques. The first technique generalizes a construction used in [1, 2] in which cubic cages of girth g are glued together using *remote vertices*, that is vertices at distance $g/2$. The second technique, which is the main content of this work, consists of constructing semi-cubic graphs of even girth using voltage graphs. With this technique, we improve the graphs given using the "identifying remote vertices" technique for girth $g = \{6, 8, 10, 12\}$. However, graphs with the same orders as those from our voltage graph construction were obtained previously for girth $g = \{6, 8\}$ (in [2,3,10]) using different techniques. Our principal contribution is for graphs of girth $g = \{10, 12\}$, where we find new graphs with orders between the lower bounds given in [3] and the upper bounds given in this paper found by identifying remote vertices.

The voltage graph construction gives us, naturally, the Heawood graph or $(3; 6)$-cage and the Tutte graph or $(3; 8)$-cage, for $m = 3$ and $g = \{6, 8\}$ respectively. These graphs occur as part of our constructions of families of $(3, m; 6)$-cages of order $4m + 2$ and $(3, m; 8)$-graphs of order $9m + 3$. We will detail how our constructions generalize the constructions given in [2, 10] in the corresponding sections. As the authors state in [2], for girth 8 and $m = \{4, 5, 6, 7\}$ these graphs, and also ours, are cages, while for the rest of the values of m they are close to the lower bound $n(3, m; 8) \geq \lceil \frac{25m}{3} \rceil + 5$ given for $m \geq 7$.

In [3], the authors proved that for m much larger than r and even girth $g \equiv 2 \mod 4$ there exist infinite families of $(r, m; g)$-graphs with few vertices, with order close to the lower bound also given in that paper. Specifically, for girth $g = 6$, the graphs described are biregular cages, because they attain the lower bound given in [13]. However, in that paper, the authors did not give an explicit construction of these graphs; they only proved their existence using a strong result about Hamiltonian graphs and girths given by Sachs in 1963 ([12]). In particular, for girth 10 the existence of a semi-regular cage continues to be open for small values of m.

For girth $g = 10$, using the identifying remote vertices technique, we obtain graphs of order greater than $22m + 2m/3$. With the voltage graph construction, we produce graphs of order $20m + 2$ for $m \geq 7$ with 2 vertices of degree m and $20m$ vertices of degree 3. This family improves the known upper bound for $(3, m; 10)$-cages and has a difference of less than $3m$ to the lower bound $n(3, m; 10) \geq \lceil \frac{53m}{3} \rceil + 9$ given in Lemma 3.4 in [3].

Finally, for girth 12, using the identifying remote vertices technique, we obtain graphs of order greater than $41m + m/3$. Using voltage graphs, we give an explicit construction of semi-cubic graphs of girth 12 using voltage graphs, giving us $(3, m; 12)$-graphs for $m \geq 10$ of order $41m + 3$ with 3 vertices of degree m and $41m$ vertices of degree 3. This family improves the upper bound and produces graphs with a difference of less than $5m$ to the lower bound $n(3, m; 12) \geq \lceil \frac{109m}{3} \rceil + 17$ given in Lemma 3.4 in [3].

As we said the techniques used in this work are basically two different:

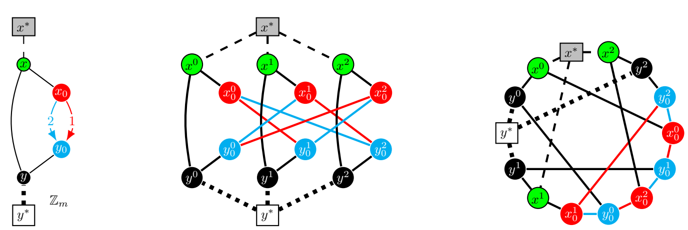
- Construct semicubic graphs identifying copies of $(r;g)$-cages: We generalize Theorem 3, given in [1], but only for cubic graphs. In this Theorem the authors identify copies of $(r;g)$-cages at *remote vertices*, which are vertices at distance at least $g/2$. These techniques are also used in [2] to construct biregular graphs of even girth.

  Theorem 2.1. *Let G be a $(3;g)$-graph of even girth and order n_g with at least two vertices at distance $g/2$. If $m = 3k + t$, we obtain $(3,m;g)$-graphs of order:*

  $$k(n_g - 2) + \begin{cases} 2 & \text{if } t = 0 \\ n_g + 2 & \text{if } t = 1 \\ n_g & \text{if } t = 2 \end{cases}$$

- Construct semi-cubic graphs using voltage graphs: A voltage graph G is a labeled directed multigraph, often including loops and parallel edges, along with a group Γ; the labels on the edges are elements of Γ (see [6, 9, 11] for standard references).

  Let Γ be a cyclic group $\mathbb{Z}_m$ with addition as the group operation. The *derived* graph G_m, also called the *lift* graph, for a voltage graph with voltage group $\mathbb{Z}_m$ is formed from G as follows: each vertex v in G corresponds to m vertices in G_m, labeled $v^0, \cdots, v^{m-1}$. An arrow in G from v to w labeled a means that vertex v^i and vertex w^{i+a} are connected by an edge in G_m, with all indices throughout the paper taken modulo m. Note that we could also have drawn an arrow from w to v labeled $-a$ and produced the same edges in the lift. If vertex v is incident with a loop labeled a in G, then in G_m, vertices v^i and v^{i+a} are incident. The collection of directed edges and loops in a voltage graph, along with their labels, is called a *voltage assignment*. An unlabeled edge in a voltage graph is assumed to have voltage assignment 0, which we also draw as undirected. In this work we introduce a "new" construction on voltage graphs, that is the notion of a *pinned vertex*, which is a special vertex of degree 1 in the voltage graph. We indicate this construction in the voltage graph using the symbol $*$ The notation $\boxed{v^*}$—(w) over $\mathbb{Z}_m$, where vertex v^* is a pinned vertex, indicates that in the lifted graph, there is a single vertex labeled v^*, which is connected to each of the vertices w^i, $i = 0, \dots, m-1$. It follows that in a derived graph when the voltage group is $\mathbb{Z}_m$, the degree of a pinned vertex is m. The figure shows an example of a voltage graph with two pinned vertices and the corresponding derived graph over $\mathbb{Z}_3$. Notice that this graph is, in fact, Heawood Graph or the $(3;8)$-cubic cage.

  Remark 2.2. *Given a voltage graph G with voltage group $\mathbb{Z}_m$ with a collection of pinned vertices $v_1^*, \dots v_s^*$ in which all vertices are degree r except the pinned vertices, which are of degree 1, the derived graph G_m is a (r,m)-biregular graph with s vertices of degree m.*

  Lemma 2.3. *If the sum of the labels in a non-reversing closed walk W in a voltage graph with voltage group $\mathbb{Z}_m$ is congruent to $0 \bmod m$, and no smaller sub-walk has voltage sum congruent to $0 \bmod m$, then the lift of W forms a cycle.*

Using the previous we construct families of semicubic graphs of girth $g \in \{6, 8, 10, 12\}$. To obtain these graphs, we describe two different families of voltage graphs called $\mathcal{G}_{4t}$, $t \geq 2$, and $\mathcal{G}_{4t+2}$ for $t \geq 1$. We use these graphs for $t \in \{2, 3\}$ in the first case and for $t \in \{1, 2\}$ in the second case to obtain families of semi-cubic graphs with girth $g = 4t$ and $g = 4t + 2$.

For more information about this work, we invite the lector to consult [4].

3 Future work

Find voltage assignments to construct graphs of girth 10 and 12 for missing values of m.

Study the same problem with an odd degree. The last two authors are making progress on this project [5].

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