Total Italian domination number in trees*

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Abstract

We provide new lower and upper bounds on the total Italian domination number of trees. In particular, we show that if T is a nontrivial tree, then the following inequality chains are satisfied.

(i) $2\gamma(T) \leq \gamma_{tI}(T) \leq n(T) - \gamma(T) + s(T)$,

(ii) $\frac{n(T)+\gamma(T)+s(T)-l(T)+1}{2} \leq \gamma_{tI}(T) \leq \frac{n(T)+\gamma(T)+l(T)}{2}$,

where $n(T)$, $\gamma(T)$, $s(T)$ and $l(T)$ are the order, the classical domination number, the number of support vertices and the number of leaves of T, respectively.

1 Introduction

Let $G(V(G), E(G))$ be a simple graph of order $n(G) = |V(G)|$ and size $m = |E(G)|$. Given a vertex v of G, $N(v) = \{x \in V(G) : xv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. The *degree* of v in G, denoted by $deg_G(v)$, is the cardinality of $N(v)$. A vertex $v \in V(G)$ is a *leaf* if $\deg_G(v) = 1$, and v is a *support* vertex if it is adjacent to a leaf. The set of leaves and support vertices are denoted by $\mathcal{L}(G)$ and $\mathcal{S}(G)$, respectively. The values $l(G)$ and $s(G)$ represent the number of leaves and the number of support vertices, respectively, i.e., $l(G) = |\mathcal{L}(G)|$ and $s(G) = |\mathcal{S}(G)|$. A set of vertices $D \subseteq V(G)$ is a *dominating set* of G if $|N(x) \cap D| \geq 1$ for every vertex $x \in V(G) \setminus D$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of G.

Recently, a new variant of domination, called *total Italian domination number*, was introduced in [3] and independently in [2], under the name of *total Roman* $\{2\}$*-domination number*. Let $f : V(G) \to \{0, 1, 2\}$ be a function defined from a connected graph G. Let $W_i = \{x \in V(G) : f(x) = i\}$ for every $i \in \{0, 1, 2\}$. The function f is called a total Italian dominating function on G if $\sum_{v\in N(x)} f(v) \geq 2$ for every vertex $x \in W_0$ and if $\sum_{v\in N(x)} f(v) \geq 1$ for every vertex $x \in W_1 \cup W_2$. The total Italian domination number of G, denoted by $\gamma_{tI}(G)$, is the minimum weight $\omega(f) = \sum_{x\in V(G)} f(x)$ among all total Italian dominating functions f on G. Further combinatorial results on total Italian domination can be found for example, in [1, 4, 6, 8].

In this work we show some new tight bounds on the total Italian domination number in trees in terms of order, domination number, number of support vertices and number of leaves of a tree.

*The results presented in this work have been published in [5].

2 Results

We begin this section with one of our main results.

Theorem 2.1. *If T is a tree of order $n(T) \geq 2$, then*

$$2\gamma(T) \leq \gamma_{tI}(T) \leq n(T) - \gamma(T) + s(T).$$

The following theorem provides two consequences derived from Theorem 2.1 and the bound $\gamma(T) \geq (n(T) - l(T) + 2)/3$ given in [7].

Theorem 2.2. *The following statements hold for any tree T of order $n(T) \geq 3$.*

(i) $\gamma(T) \leq \frac{n(T)+s(T)}{3}$.

(ii) $\gamma_{tI}(T) \leq \frac{2n(T)+l(T)+3s(T)-2}{3}$.

We say that a tree T belongs to the family $\mathcal{T}$ if it satisfies one of the following two conditions.

- T is a subdivided star.
- T can be obtained from a star $K_{1,n}$ by subdividing exactly $n-1$ edges at most twice.

The following theorem provides a lower bound for any tree T with $s(T) = l(T)$. In addition, this result shows that the lower bound given in Theorem 2.1 is achieved for any tree T belongs to the family $\mathcal{T}$ previously defined.

Theorem 2.3. *The following statements hold for any tree T of order $n(T) \geq 2$ with $s(T) = l(T)$.*

(i) $\gamma_{tI}(T) \geq \gamma(T) + \Delta(T)$.

(ii) $\gamma_{tI}(T) = \gamma(T) + \Delta(T)$ *if and only if* $T \in \mathcal{T}$.

Finally, we show our second main result obtained. Highlight that the bounds obtained in this inequality chain are tight.

Theorem 2.4. *If T is a tree of order $n(T) \geq 2$, then*

$$\frac{n(T) + \gamma(T) + s(T) - l(T) + 1}{2} \leq \gamma_{tI}(T) \leq \frac{n(T) + \gamma(T) + l(T)}{2}.$$

3 Conclusions and open problems

In this work we obtained new lower and upper bounds on the total Italian domination number in terms of order, domination number, number of support vertices and number of leaves of a nontrivial tree. In addition, we discussed some extreme cases. Finally, we propose some open problems that arise from the results obtained.

(a) Characterize the trees attaining the bounds given in Theorems 2.1, 2.2 and 2.4.

(b) Obtain new lower and upper bounds on the total Italian domination number of trees.

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