Some results of the Total Triple Roman Domination in Graphs

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Abstract

In this paper we begin the study of the total triple roman domination in which every vertex is protected by 3 legions and each vertex with positive label has, at least, one neighbor with positive label. The total triple roman dominance is characterized by labeling the vertices of the graph in the set $\{0, 1, 2, 3, 4\}$ to guarantee that $f(AN[v]) \geq |AN(v)| + 3$, where $AN(v)$ represents the active neighbors of vertex v, that is, those with a positive label. Higher bounds have been tested based on the order and the maximum and minimum degrees of the graph for the total triple roman domination, verifying that said bounds are reached,the exact value of the parameter for some families of graphs has been obtained and also, the models for the formulation of integer linear programming to solve triple Roman domination problems are implemented by adding more conditions in order to be able to solve total triple Roman domination problems in the most efficient way.

1 Introduction

Total triple Roman domination in Grafos emerges as a variant of the triple Roman domination introduced by Ahangar et al. [1] in 2021. Amjadi et al. [2] in 2021, also introduce some variations of Roman domination. The concept of Roman domination was established in 2004 by Cockaine et al. [4] basing himself on the defensive strategy of the Roman Empire reported in the articles by ReVelle and Rosing [5] and Stewart [6]. Roman domination emerges as a solution to a classic problem of military defensive strategy. In order to understand the concept, we must start from the idea that Emperor Constantine I the Great conceived to protect the vast empire from him. The emperor decreed that any city could be defended by a legion from a neighboring city on the condition that the latter did not remain defenseless. This translates to a mathematical model in which we have a finite graph whose nodes can have labels $\{0, 1, 2\}$, provided that each vertex with label 0 must have at least one neighbor with a 2. The objective is to minimize the sum of the labels, which, translated into the defensive problem, consists of having a minimum presence of legions that defend the territories with the established conditions. The triple Roman domination is presented with the objective of having each territory defended by three legions minimizing its cost. Consider f as a function $f : V(G) \to \{0,1,2,3,4\}$ on the graph $G = (V,E)$, such that, $f(AN[v]) \geq |AN(v)| + 3$ for any $v \in V$ with $f(v) < 3$, where $AN(v) \subseteq V$ is the set of all vertices adjacent to v with positive label. The total triple Roman domination, was born as a new variant of the triple Roman domination with the purpose before an individual attack to the nodes to make it more efficient. This variant defined by a function f on the graph G must satisfy the above conditions for triple Roman domination, in addition to any subgraph induced in G by the set $u \in V$, such that $f(u) \neq 0$ does not have isolated vertices. The number of total triple Roman domination $\gamma_{[t3R]}\ (G)$ is defined as the minimum of the weight or sum of the labels of the vertices $w(f) = \sum f(v)$ and the function f of minimum weight defined in G is defined as a $\gamma_{[t3R]}\ (G)-$function 'on. In this work some heights are established, graphs that reach them and exact values are studied for some families of graphs such as paths, cycles, bistar, complete bipartite and spider graphs. Also, the models for the formulation of integer linear programming proposed by Vengaldas et al. [7] to solve triple Roman domination problems by adding more conditions in order to be able to solve total triple Roman domination problems in the most efficient way.

2 Results

In this section we present some of the most relevant results studied about bounds and exact values that have been obtained.

Proposition 2.1. *Let G be an ntc-graph of order n, $\delta \geq 2$, girth ≥ 5 and maximun degree Δ. Then $\gamma_{[t3R]}\ (G) \leq 2(n - \Delta + 1)$.*

This bound is matched in a P_6.

Proposition 2.2. *Let G be an r-regular graph of order n and girth ≥ 7. Then $\gamma_{[t3R]}\ (G) \leq 2n - 2r^2 + 3r - 1$.*

It is easy to check that the last bound is the exact value of $\gamma_{[t3R]}(C_7)$.

Proposition 2.3. *Let G be an ntc-graph then $\gamma_{[t3R]}\ (G) \geq \left\lceil \frac{2n+(\Delta-1)\gamma}{\Delta} \right\rceil$.*

Proposition 2.4. *Let G be an ntc-graph of order $n \geq 2$ and maximum degree $\Delta \geq 3$. Then,*

$$\gamma_{[t3R]}(G) \geq \left\lceil \frac{4n}{\Delta(G)+1} \right\rceil$$

Lemma 2.5. *Let G be an ntc-graph of order n, $\Delta \leq 2$ and let f be a $\gamma_{[t3R]}\ (G)$-function such that the number of vertices assigned 0 under f is minimum. Then $f(x) \neq 4$ for each $x \in V(G)$.*

Lemma 2.6. *Let G be a connected graph with $\Delta \leq 2$ and $v_{i_0-2}, v_{i_0-1}, v_{i_0}, v_{i_0+1}, v_{i_0+2}$ be consecutive vertices of G. Then there exists a $\gamma_{[t3R]}\ (G)$-function g such that $0 \in \{g(v_j),\ i_0 - 2 \leq j \leq i_0 + 2\}$.*

Thanks to this lemma and others that allow us to relabel the vertices, they have allowed the demonstration of the following results.

Proposition 2.7. *Let G be any connected graph with $\Delta \leq 2$, $n \geq 5$ and let f be a $\gamma_{[t3R]}\ (G)$-function such that the number of vertices assigned 0 under f is minimum. Then*

$$\left\lceil \frac{6n}{5} \right\rceil + 2 \leq \gamma_{[t3R]}\ (G) \leq \left\lceil \frac{8n}{5} \right\rceil.$$

Proposition 2.8. *Let be $n \geq 3$ a positive integer. let $f = (V_0, V_1, V_2, V_3, \emptyset)$ be a $\gamma_{[t3R]}(P_n)$-function on P_n, such that the number of vertices assigned 0 under f is minimum. Then*

$$\gamma_{[t3R]}(P_n) = \begin{cases} 8 & \text{if } n = 4 \\ \lceil \frac{3n}{2} \rceil & \text{if } n \equiv 0,1,3,5,7 \ (mod\ 8) \\ & \quad n \neq 7 \\ \lceil \frac{3n}{2} \rceil + 1 & \text{if } n \equiv 2,4,6 \ (mod\ 8) \\ & \quad n = 7 \\ & \quad n \neq 4 \end{cases}$$

Proposition 2.9. *Let be $n \geq 3$ a positive integer. let $f = (V_0, V_1, V_2, V_3, \emptyset)$ be a $\gamma_{[t3R]}(C_n)$-function on C_n, such that the number of vertices assigned 0 under f is minimum. Then*

$$\gamma_{[t3R]}(C_n) = \begin{cases} \lceil \frac{3n}{2} \rceil & \text{if } n \equiv 0,1,3,5,7 \ (mod\ 8) \\ \lceil \frac{3n}{2} \rceil + 1 & \text{if } n \equiv 2,4,6 \ (mod\ 8) \end{cases}$$

Once the exact value for paths and cycles is known, we observe that the lower bound 2.7 is matched for P_5, P_6, P_8, C_5, C_6, C_7 and C_8, and the upper bound 2.7 is matched for P_3, P_5, P_6, P_7,C_3,C_4, C_5, C_6, P_{10} and C_{10}.

Proposition 2.10. *Let p,q be positive integers, then*

$\gamma_{[t3R]}(K_{1,1}) = 4$, $\quad \gamma_{[t3R]}(K_{1,q}) = 5$, $\ forq \geq 2$, $\quad \gamma_{[t3R]}(K_{2,q}) = 7$, $\ forq \geq 2$, $\quad \gamma_{[t3R]}(K_{p,q}) = 8$, $\ forp, q \geq 3$

Proposition 2.11. *Let be p,q positive integers, then*

$\gamma_{[t3R]}(S_{p,q}) = 8$, *for* $p, q \geq 1$

Proposition 2.12. *Let $0 \leq q \leq p$ be positive integers.*

1. *If $p = q \geq 3$, $SP_{p,q}$ is a healthy spider then $\gamma_{[t3R]}(SP_{p,q}) = 4p$*
2. *If $q < p$ then $\gamma_{[t3R]}(SP_{p,q}) = 4 + 4q$*

We denote ILPTTRDP a integer lineal programming formulations for total triple roman domination problems. First of all, we will define the following variables for $u \in V$:

$$a_u = \begin{cases} 1 & \text{if } f(u) = 1. \\ 0 & otherwise. \end{cases} \quad b_u = \begin{cases} 1 & \text{if } f(u) = 2. \\ 0 & otherwise. \end{cases} \quad c_u = \begin{cases} 1 & \text{if } f(u) = 3. \\ 0 & otherwise. \end{cases}$$

$$d_u = \begin{cases} 1 & \text{if } f(u) = 4. \\ 0 & otherwise. \end{cases} \quad e_u = \begin{cases} 1 & \text{if } \sum_{t \in N(u)} b_t \geq 1. \\ 0 & otherwise. \end{cases} \quad f_u = \begin{cases} 1 & \text{if } \sum_{t \in N(u)} c_t \geq 1. \\ 0 & otherwise. \end{cases}$$

In our case, the ILP that we propose will be to calculate the minimum of the weight of the function f, that is (ILPTTRDP1)

$$\min \left\{ \sum_{u \in V} a_u + 2 \sum_{u \in V} b_u + 3 \sum_{u \in V} c_u + 4 \sum_{u \in V} d_u \right\},$$

With the constraints,

$$a_u + b_u + c_u + d_u + \sum_{t\in N(u)} d_t + \frac{1}{3}(\sum_{t\in N(u)} b_t) + \frac{1}{2}(e_u + f_u) \geq 1,$$

$$\frac{1}{2}(\sum_{t\in N(u)} b_t) + \sum_{t\in N(u)} (c_t + d_t) \geq a_u, \qquad \sum_{t\in N(u)} (b_t + c_t + d_t) \geq b_u,$$

$$a_u + b_u + c_u + d_u \leq 1, \quad a_u, b_u, c_u, d_u, e_u, f_u \in \{0,1\}, \quad \sum_{t\in N(u)} (a_t + b_t + c_t + d_t) \geq 1$$

As a consequence, we have a total of $4|V|$ variables and $6|V|$ constraints. As already demonstrated Vengaldas et al. [7], the first six restrictions correspond to the mathematical formulation of a triple domination problem. And the last restriction , refers to the imposition that every vertex must have at least one adjacent neighbor with a positive label, therefore the problem of total triple Roman domination would be formulated.

3 Conclusions

Finding the total triple Roman domination number is a difficult task for the different families of graphs, we have started by studying their bounds, obtaining lower and upper bounds and graphs that reach them. In particular, the exact value for this variant has been found for some families of graphs. And finally, given the difficulty involved in studying the variants of Roman domination that increase the defenses in their nodes, a new restriction has been proposed for its integer linear programming that efficiently solves the problems of total triple Roman domination.

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