Domination and matching numbers for power hypergraphs ⓢ

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Abstract.

We present some examples that refute two theorems concerning the equality of the domination and matching numbers for hypergraphs.

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1 Introduction

A *(finite) hypergraph* $\mathcal{H} = (V, E)$ consists of a (finite) set V and a collection E of non-empty subsets of V. The elements of V are called *vertices* and the elements of E are called *hyperedges*, or symply *edges* of the hypergraph. A $k-$*uniform* hypergraph is a hypergraph such that each edge consists of k vertices. A simple graph with no isolated vertices is a 2-uniform hypergraph. Two vertices of $\mathcal{H}$, u and v, are *adjacent* if there is an edge e such that $u, v, \in e$.

Given a hypergraph $\mathcal{H} = (V, E)$, $D \subset V$ is a *dominating set* of $\mathcal{H}$ if for every $v \in V - D$ there exists $u \in D$ such that u and v are adjacent. The minimum cardinality of a dominating set of $\mathcal{H}$, $\gamma(\mathcal{H})$, is its *dominating number.* A *matching* in $\mathcal{H}$ is a set of disjoint hyperedges. The *matching number* of $\mathcal{H}$, $\nu(\mathcal{H})$, is the maximum size of a matching in $\mathcal{H}$. A subset $T \subset V$ is a *transversal* (or a *vertex cover*) of $\mathcal{H}$ if T has nonempty intersection with every hyperedge of $\mathcal{H}$. The *transversal number* of $\mathcal{H}$, $\tau(\mathcal{H})$, is the minimum size of a transversal of $\mathcal{H}$.

Definition 1. ([2]) *Let $G = (V, E)$ be a simple graph. For any $k \geq 3$ and $1 \leq s \leq \frac{k}{2}$, the* generalized power *of G, denoted by $G^{k,s}$, is defined as the k-uniform hypergraph with the vertex set $\{\mathbf{v} : v \in V\} \cup \{\mathbf{e} : e \in E\}$, and the edge set $\{\mathbf{u} \cup \mathbf{v} \cup \mathbf{e} : e = \{u, v\} \in E\}$, where $\mathbf{v}$ is a s-set containing v and $\mathbf{e}$ is a $(k - 2s)$-set corresponding to e. $G^{k,s}$ is called the* generalized power hypergraph *obtained from G. Particularly, for $s = 1$, $G^{k,1}$ is the kth-*power hypergraph *of G.*

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Intuitively, $G^{k,s}$ is obtained from G by replacing each vertex v by an s-subset $\mathbf{v}$ and each edge by a k-set obtained from $\mathbf{v} \cup \mathbf{u}$ by adding $(k-2s)$ new vertices; see Figure 1 for illustration.

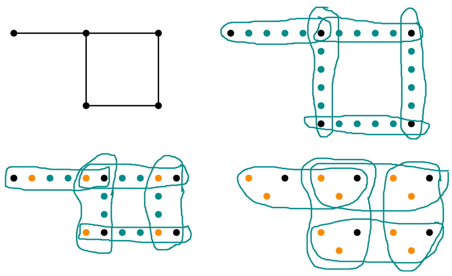

Figure 1.1: The graph G, the 6th-power hypergraph $G^{6,1}$, the generalized power hypergraph $G^{6,2}$, and the generalized power hypergraph $G^{6,3}$.

Note 1. *From now on, $G^{k,s}$ will denote the generalized power hypergraph obtained from the graph G and we will call it power hypergraph, for the sake of simplicity.*

It is not difficult to check that the domination number is not hereditary for power hypergraphs in general. However, the next result holds.

Proposition 1. ([1]) *Let $G = (V, E)$ be a simple graph, for $k \geq 3$, $1 \leq s \leq k/2$ natural numbers, we get:*

11. $\nu(G^{k,s}) = \nu(G)$ *and* $\tau(G^{k,s}) = \tau(G)$.

12. $\gamma(G^{k,\frac{k}{2}}) = \gamma(G)$.

13. $\gamma(G^{k,s}) = \tau(G^{k,s})$, *if* $1 \leq s < k/2$.

Theorem 1. ([1]) *For any connected generalized power hypergraph $G^{k,s}$ and $k \geq 3$, if $1 \leq s < \frac{k}{2}$, we have $\nu(G^{k,s}) \leq \gamma(G^{k,s}) \leq 2\nu(G^{k,s})$ and if $s = \frac{k}{2}$, we get $\gamma(G^{k,s}) \leq \nu(G^{k,s})$.*

2 Main results

Some extremal hypergraphs for the bounds of Theorem 1 are collected in [1]. Particularly, the authors define the family of power hypergraphs obtained from bipartite graphs, for which domination and matching numbers coincide. Theorems 2.2 and 3.4 of [1] affirm that this family characterizes those extremal hypergraphs. However, we refute these theorems by presenting some counterexamples for those assertions.

Example 1. *Let $G = C_p \vee C_q$ be the wedge of two cycles joined by a common vertex, $p, q \geq 3$, being p or q an odd number. The following identities hold.*

- $\nu(G) = \lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor$; *set* $n := \nu(G)$.
- $\tau(G) = \lceil \frac{p}{2} \rceil + \lceil \frac{q}{2} \rceil - 1$

From Proposition 1, we get $\gamma(G^{k,s}) = \tau(G^{k,s}) = \tau(G)$ *for* $1 \leq s < k/2$.

21. *If p and q are both odd numbers, then* $\lceil \frac{p}{2} \rceil + \lceil \frac{q}{2} \rceil - 1 = n + 1$ *and, therefore*

$$\gamma(G^{k,s}) = n + 1 = \nu(G) + 1 = \nu(G^{k,s}) + 1.$$

22. *If p is even and q is odd, then* $\lceil \frac{p}{2} \rceil + \lceil \frac{q}{2} \rceil - 1 = n$, *and hence*

$$\gamma(G^{k,s}) = n = \nu(G^{k,s}).$$

Let us notice that if p and q are both even, then G is bipartite and $G^{k,s}$ belongs to the family described in [1] having $\gamma(G^{k,s}) = \nu(G^{k,s})$.

Observe that the family of graphs of item 22 are not bipartite and the corresponding power hypergraphs are extremal for the equality of dominating and matching numbers. This is a first counterexample of Theorems 2.2 and 3.4 of [1], but we guess there are more of them.

In despite of previous facts, our goal is to describe families of non-extremal power hypergraphs, which are those ones whose domination number does not reach those extremal values given by Theorem 1. More precisely, we want to find power hypergraphs whose domination number fills the gap between its matching number and twice its matching number. Analogously, we also look for power hypergraphs with domination number smaller than its matching number. We start by studying some very well known graphs.

Lemma 1. *The following statements hold.*

21. *For any odd cycle $G = C_{2l+1}$, we get* $\gamma(G^{k,s}) = \nu(G^{k,s}) + 1 = l + 1$, *for any* $l \geq 2$, $1 \leq s < \frac{k}{2}$.

22. *For any $n \geq 3$ the wheel graph $G = W_n = C_n + K_1$ verifies*

    a) $\gamma(G^{k,s}) = \nu(G^{k,s}) + 1 = l + 1$, *for any* $1 \leq s < \frac{k}{2}$.

    b) $1 = \gamma(G^{k,\frac{k}{2}}) < \nu(G^{k,\frac{k}{2}}) = \lceil \frac{n}{2} \rceil$.

As a consequence, we obtain a first general result.

Proposition 2. *For any integer* $n \geq 2$ *there exists a graph* G *such that* $\nu(G^{k,s}) = n$ *and* $\gamma(G^{k,s}) = \nu(G^{k,s}) + 1$*, for any* k*,* $1 \leq s < \frac{k}{2}$.

Other examples providing $\gamma(G^{k,s}) = \nu(G^{k,s}) + 1$ (for any k, $1 \leq s < \frac{k}{2}$) are Petersen's graph and the graph of the octahedron.

The graph of the dodecahedron leads to power hypergraphs $G^{k,s}$ so that $\gamma(G^{k,s}) = \nu(G^{k,s}) + 2 = 12$, for $1 \leq s < \frac{k}{2}$, , and $\gamma(G^{k,\frac{k}{2}}) = 6$. The graph of the icosahedron produces power hypergraphs $G^{k,s}$ with $\gamma(G^{k,s}) = \nu(G^{k,s}) + 3 = 2\nu(G^{k,s}) - 3 = 9$, for $1 \leq s < \frac{k}{2}$ and $\gamma(G^{k,\frac{k}{2}}) = 2$.

Concerning the other extremal value, $2\nu - 1$, we present the following

Proposition 3. *For any integer* $n \geq 2$ *there exist graphs* G_1 *and* G_2 *such that* $\nu(G_1^{k,s}) = n$ *and* $\gamma(G_1^{k,s}) = 2\nu(G_1^{k,s}) - 1$*, for any* $k, 1 \leq s < \frac{k}{2}$*; and* $\gamma(G_2^{k,\frac{k}{2}}) < \nu(G_2^{k,\frac{k}{2}})$.

The proof derives from the case of even complete graphs $G = K_{2l}$ with $l \geq 2$. In fact, we get $\gamma(G^{k,s}) = 2\nu(G^{k,s}) - 1 = 2l - 1$, for any $1 \leq s < \frac{k}{2}$; and $1 = \gamma(G^{k,\frac{k}{2}}) < \nu(G^{k,\frac{k}{2}}) = l$.

Problems:

21. Finding the complete family of graphs G whose power hypergraphs $G^{k,s}$ verify one of the following equalities, for any k, $1 \leq s < \frac{k}{2}$

$$\gamma(G^{k,s}) = \nu(G^{k,s}) + 1 \text{ or } \gamma(G^{k,s}) = 2\nu(G^{k,s}) - 1.$$

22. For each integer $n \geq 2$, and every $m \in \{n+1, \ldots, 2n-1\}$, does exist a graph whose power hypergraph verifies $\nu(G^{k,s}) = n$ and $\gamma(G^{k,s}) = m$, for $1 \leq s < \frac{k}{2}$?

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