Graphs attaining the upper bound on the isolation number[S]

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Abstract. A set D of vertices of a graph is isolating if the set of vertices not belonging to D or not having a neighbour in D is independent. The isolation number of a graph G, denoted by $\iota(G)$, is the minimum cardinality of an isolating set. It is known that $\iota(G) \leq n/3$ if G is a connected graph of order n, $n \geq 3$, different from a cycle of order 5. Moreover, trees attaining this upper bound have been characterized. In this work we characterize all unicyclic and block graphs of order n such that $\iota(G) = n/3$.

Keywords. *domination, isolation, unicyclic graph, block graph*

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1 Introduction

Domination in graphs has deserved a lot of attention since it was introduced in the fifties motivated by chessboard problems, among others. A set D of vertices of a graph G is *dominating* if every vertex not in D has at least one neighbour in D. The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. There is an extensive literature on dominating sets in graphs, see for example the book [4] and references therein. The definition of dominating set can be reformulated as follows. For every graph $G = (V, E)$, let $N[v] = \{v\} \cup \{u \in V : \ uv \in E\}$ be the *closed neighborhood* of the vertex $v \in V$. If $S \subseteq V$, then $N[S] = \cup_{v \in S} N[v]$. With this terminology, D is a dominating set of G if and only if $V = N[D]$. The concept of isolation arises by relaxing this condition. Concretely, a set D of vertices of a graph G is *isolating* if the set of

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vertices not in $N[D]$ is independent [1]. The *isolation number* of G, denoted by $\iota(G)$, is the minimum cardinality of an isolating set. The following upper bound for the isolation number has been proven.

Theorem 1. *[1] Let G be a connected graph on $n \geq 3$ vertices different from C_5. Then $\iota(G) \leq n/3$ and this bound is sharp.*

Our goal is to characterize all the graphs attaining this upper bound. Trees are characterized in [2], and in this work we characterize unicyclic graphs and block graphs. Proofs are omitted because of limited space.

For the case of unicyclic graphs, the characterization is closely related to that of trees. We recall here such trees. Let $\mathcal{T}$ be the family of trees T that can be obtained from a sequence of trees $T_1, \ldots, T_j$, $j \geq 1$, such that T_1 is a path P_3; $T = T_j$; and, if $1 \leq i \leq j-1$, then T_{i+1} can be obtained from T_i by adding a path P_3 and an edge xy, where x is a vertex at a distance two from a leaf of T_i and y is a leaf of the path P_3.

Theorem 2. *[2] If T is a tree of order n, then $\iota(G) = n/3$ if and only if $T \in \mathcal{T}$.*

The family $\mathcal{T}$ can also be described as follows. A tree T belongs to $\mathcal{T}$ if and only if it can be obtained by attaching exactly one copy of a path P_3 at every vertex v of a tree T_0 by identifying v with a leaf of P_3 (see an example in Figure 1.1a). It is easy to see that every minimum isolating set of T contains at least one vertex of each P_3. This fact suggests the following construction, that

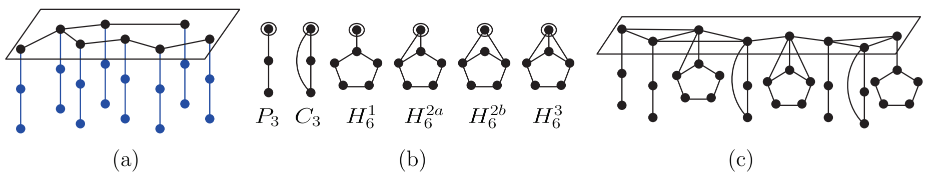

Figure 1.1: (a) A tree of the family $\mathcal{T}$; (b) the graphs P_3, C_3, H_6^1, H_6^{2a}, H_6^{2b} and H_6^3; (c) a graph of $\mathcal{G}$.

provides a family of graphs attaining the upper bound on the isolation number. Let P_3, C_3, H_6^1, H_6^{2a}, H_6^{2b} and H_6^3 denote the graphs depicted in Figure 1.1b. The family $\mathcal{G}$ consists of all graphs obtained by attaching exactly one copy of one of the graphs P_3, C_3, H_6^1, H_6^{2a}, H_6^{2b} or H_6^3 at every vertex v of a given graph G_0, by identifying v with the circled vertex of the attached graph (see an example in Figure 1.1c). Since every minimum isolating set of a graph belonging to $\mathcal{G}$ contains at least one vertex of each attached graph, if it is P_3 or C_3, and at least

2 vertices of each attached graph in the remaining cases, the following result holds.

Proposition 1. *If G is a graph of order n that belongs to $\mathcal{G}$, then $\iota(G) = n/3$.*

2 Unicyclic graphs

A connected graph G is *unicyclic* if it contains exactly one cycle. The cycles C_6 and C_9 are unicyclic graphs attaining the upper bound on the isolation number. Let $\mathcal{U}$ denote the family of unicyclic graphs belonging to $\mathcal{G}$. By Proposition 1, these graphs attain also the upper bound. Notice that these graphs can be obtained by attaching either a copy of P_3 at every vertex of a unicyclic graph; or a copy of C_3 at a vertex of a tree T and a copy of P_3 at any other vertex of T; or a copy of H_6^1 at a vertex of a tree T and a copy of P_3 at any other vertex of T (see Figure 1.2).

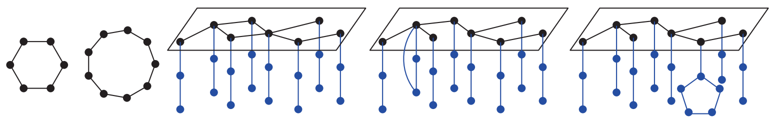

Figure 1.2: Unicyclic graphs attaining the upper bound on the isolation number: C_6, C_9 and unicyclic graphs of the family $\mathcal{G}$.

Theorem 3. *If G is a unicyclic graph of order n, then $\iota(G) = n/3$ if and only if $G \in \{C_6, C_9\} \cup \mathcal{U}$.*

3 Block graphs

A vertex v of a connected graph G is a *cut vertex* if the removal of v from G results in a disconnected graph. A *block* of a graph G is a maximal connected subgraph of G without cut vertices. A connected graph G is a *block graph* if every block of G is a complete graph. Let $\mathcal{B}$ denote the family of block graphs belonging to $\mathcal{G}$. Notice that $\mathcal{B}$ contains the graphs obtained by attaching P_3 or C_3 at every vertex of a block graph (see Figure 1.3).

Theorem 4. *If G is a block graph, then $\iota(G) = n/3$ if and only if $G \in \mathcal{B}$.*

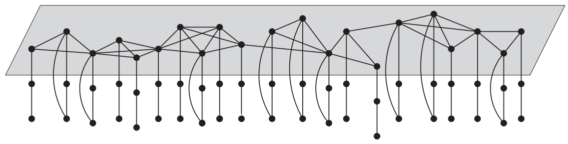

Figure 1.3: A graph of the family $\mathcal{B}$.

4 Concluding remarks

In some sense, the obtained results have the same flavour as the characterization of graphs attaining the upper bound on the domination number. Concretely, it is well known that a graph G of order n without isolated vertices has domination number at most $n/2$ [5], and the graphs attaining this bound are the cycle C_4 and the corona graphs $G \circ K_1$ [3, 6], obtained by attaching a copy of K_2 at every vertex of G by identifying the vertex of G with a vertex of K_2.

For the isolation number, every graph in $\mathcal{G}$ attains the upper bound, and these graphs can be obtained by attaching a copy of P_3, C_3, H_6^1, H_6^{2a}, H_6^{2b} or H_6^3 at every vertex of a graph G. At the moment, for the families of trees and block graphs, all graphs attaining the upper bound are in $\mathcal{G}$. For unicyclic graphs, C_6 and C_9 are the only graphs not in $\mathcal{G}$ attaining the upper bound. It remains to characterize the graphs not in $\mathcal{G}$ attaining the upper bound for general graphs.

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