ROUDNEFF'S CONJECTURE IN DIMENSION 4

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Abstract

J.-P. Roudneff conjectured in 1991 that every arrangement of $n \geq 2d+1 \geq 5$ (pseudo) hyperplanes in the real projective space $\mathbb{P}^d$ has at most $\sum_{i=0}^{d-2}\binom{n-1}{i}$ cells bounded by each hyperplane. The conjecture is true for $d = 2, 3$. The main result of this manuscript, is to show the validity of this conjecture for $d = 4$.

Keywords: Roudneff's conjecture, Oriented Matroids, Arrangements of Hyperplanes.

1 Introduction

An *Euclidean* (resp. *projective*) d-arrangement of n hyperplanes $H(d,n)$ is a finite collection of hyperplanes in the Euclidean space $\mathbb{R}^d$ (resp. the real projective space $\mathbb{P}^d$) such that no point belongs to every hyperplane of $H(d,n)$. Any arrangement $H(d,n)$ decomposes $\mathbb{R}^d$ (resp. $\mathbb{P}^d$) into a d-dimensional cell complex. It is clear that any d-cell c of $H(d,n)$ has at most n *facets* (that is, $(d-1)$-cells). We say that c is a *complete cell* of $H(d,n)$ if c has exactly n facets, i.e., c is bounded by each hyperplane of $H(d,n)$.

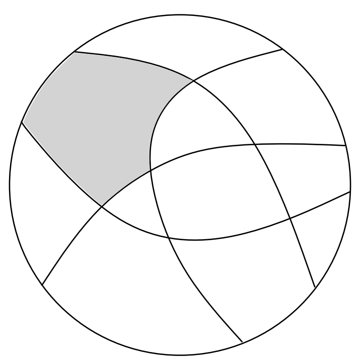

Figure 1: An arrangement of 5 hyperplanes in $\mathbb{P}^2$. The gray cell is a complete cell.

The *cyclic polytope* of dimension d with n vertices, discovered by Carathéodory [2], is the convex hull in $\mathbb{R}^d$ of $n \geq d+1 \geq 3$ different points $x(t_1), \ldots, x(t_n)$ of the moment curve $x : \mathbb{R} \longrightarrow \mathbb{R}^d,\ t \mapsto (t, t^2, \ldots, t^d)$. Cyclic polytopes play an important role in the combinatorial convex geometry due to their connection with certain extremal problems, For example, the Upper Bound theorem due to McMullen [8]. *Cyclic arrangements* are defined as the dual of the cyclic polytopes. As for cyclic polytopes, cyclic arrangements also have extremal properties. For instance, Shannon [9] has introduced cyclic arrangements on dimension d as examples of projective arrangements with a minimum number of cells with $(d+1)$ facets.

Let denote by $C_d(n)$ the number of complete cells of the cyclic arrangements on dimension d with n hyperplanes. Roudneff [6] proved that $C_d(n) \geq \sum_{i=0}^{d-2} \binom{n-1}{i}$ and that is tight for all $n \geq 2d+1$. Moreover, he conjectured that in that case, cyclic arrangements have the maximum number of complete cells.

Conjecture 1.1. *[6, Conjecture 2.2] Every arrangement of $n \geq 2d+1 \geq 5$ (pseudo) hyperplanes in $\mathbb{P}^d$ has at most $C_d(n)$ complete cells.*

The conjecture is true for $d = 2$ (that is, any arrangement of n pseudolines in $\mathbb{P}^2$ contains at most one complete cell), Ramírez Alfonsín [5] proved the case $d = 3$ and in [7] the authors proved it for arrangements arising from Lawrence oriented matroids.

In [3] calculated the exact number of complete cells of cyclic arrangements for any positive integers d and n with $n \geq d+1$, namely,

$$C_d(n) = \binom{d}{n-d} + \binom{d-1}{n-d-1} + \sum_{i=0}^{d-2} \binom{n-1}{i}. \tag{1}$$

Thus, in view of Roudneff's conjecture, Montejano and Ramírez Alfonsín [7] asked the following.

Question 1.2. *Is it true that every (pseudo) arrangement of $n \geq d+1 \geq 3$ hyperplanes in $\mathbb{P}^d$ has at most $C_d(n)$ complete cells?*

The main result of this work is to answer affirmatively to Question 1.2 for $d = 4$. As a consequence, we prove Roudneff's conjecture for dimension 4, giving more credit to the general conjecture. Moreover, with some simple observations, we can finish answering to Question 1.2 for $d = 3$.

2 Results

Many of the combinatorial properties of arrangements of (pseudo) hyperplanes can be studied in the language of oriented matroids. The so-called Topological Representation Theorem, due to Folkman and Lawrence [4], states that the acyclic reorientation classes of oriented matroids on n elements and rank r (without loops or parallel elements) are in one-to-one correspondence with the classes of isomorphism of arrangements of n (pseudo) hyperplanes in $\mathbb{P}^{r-1}$.

An arrangement $H(d, n)$ is called *simple* if $n \geq d$ and every intersection of d pseudo-hyperplanes is a unique distinct point. It is known that simple arrangements correspond to uniform oriented matroids. The d-cells of any arrangement $H(d, n)$ are usually called topes since they are in one-to-one correspondence with the topes of the oriented matroids M on n elements of rank $r = d+1$ of its corresponding acyclic reorientation class. It is known that a tope is a complete cell if reorienting any single element, the resulting sign-vector is also a tope.

Theorem 2.1. *Each of the 135 acyclic reorientation classes of uniform rank 5 oriented matroids on 8 elements have at most $2C_4(8)$ complete cells. Moreover, there is only 1 acyclic reorientation class with exactly $2C_4(8)$ complete cells.*

Theorem 2.2. *Each of the* 9 276 595 *acyclic reorientation classes of uniform rank* 5 *oriented matroids on* 9 *elements have at most* $2C_4(9)$ *complete cells. Moreover, the class of the alternating oriented matroid is the only one with exactly* $2C_4(9)$ *complete cells.*

Theorem 2.3. *Every arrangement of* $n \geq 5$ *(pseudo) hyperplanes in* $\mathbb{P}^4$ *has at most* $C_4(n)$ *complete cells.*

3 Conclusions

Until now Roudneff's conjecture has been verified for dimensions $d = 2, 3, 4$ and for arrangements arising from Lawrence oriented matroids. To prove Roudneff's conjecture for dimension 4, we used some ideas and techniques taken from oriented matroids.

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