Mutual-Visibility and Total Mutual-Visibility in Hamming Graphs

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If X is a subset of vertices of a graph G, then vertices u and v are X-visible if there exists a shortest u,v-path P such that $V(P) \cap X \subseteq \{u,v\}$. If each two vertices from X are X-visible, then X is a *mutual-visibility set*. The concept of mutual-visibility in graphs was introduced to the literature by Di Stefano in [4]. Further, X is a *total mutual-visibility set* of G if every pair of vertices of G is X-visible. The cardinality of a largest (total) mutual-visibility set of G is the *(total) mutual-visibility number* and denoted by $(\mu_{\rm t}(G))$ $\mu(G)$.

In this talk, we will present the main developments in this field so far. In the seminal paper [4] the mutual-visibility number was investigated on several classes of graphs, after showing that the problem of finding the mutual-visibility set with a size larger than a given number is NP-complete. In [2], the mutual-visibility problem was studied on Cartesian products and on triangle-free graphs, while in [3] the focus was on strong products. During the later investigation, the concept of total mutual-visibility has proved to be a natural and necessary tool for its exploration. Several results on the total mutual-visibility number are obtained for Cartesian product graphs in [5, 6]. Especially, the total mutual-visibility number was determined for the Cartesian product of a tree and an arbitrary graph [6]. In [5], Kuziak and Rodríguez-Velázquez extended several of the results from [6]. For instance, if G is a graph in which the set of simplicial vertices is independent and form a largest total mutual-visibility set, then for any graph H we have $\mu_{\rm t}(G \,\Box\, H) = \mu_{\rm t}(G)\, \mu_{\rm t}(H)$.

The intrinsic difficulty of the (total) mutual-visibility turns out very nicely when investigating it on Cartesian products of complete graphs which are also known as the Hamming graphs. For this sake recall the famous Zarankiewicz's Problem which asks for the following. If m, n, s, and t are given positive integers, then determine the maximum number of 1s that an $m \times n$ binary matrix can have provided that it contains no constant $s \times t$ submatrix of 1s. Denoting this number by $z(m,n;s,t)$, we have the following result.

Theorem 1 ([2]) *If $m,n \geq 2$, then $\mu(K_m \,\Box\, K_n) = z(m,n;2,2)$.*

For the total mutual-visibility number of Hamming graphs with two factors we have:

Proposition 2 ([6]) *If $n, m \geq 2$, then $\mu_{\rm t}(K_n \,\square\, K_m) = \max\{n, m\}$.*

On the the other hand, the total mutual-visibility number of Hamming graphs with more that two factors is a challenging problem. Among other related results, the following results will be presented in the talk.

Theorem 3 ([6]) *Let G be a Hamming graph. Then $X \subseteq V(G)$ is a total mutual-visibility set of G if and only if $2 \notin \{d_G(u, v) : \; u, v \in X\}$.*

Theorem 4 ([6]) *If $r \geq 3$, then*

$$\mu_{\rm t}(K_{n_1} \,\square\, \cdots \,\square\, K_{n_r}) = O(n^{r-2}) \,,$$

where $n = \sum_{i=1}^{r} n_i$.

Theorem 5 *For every two positive integers s and $r \geq 3$, it holds that*

$$\mu_{\rm t}(K_s^{\square, r}) = \Theta(s^{r-2}) \,.$$

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